ON SYSTEMS OF RATIONAL DIFFERENCE EQUATIONS AND PERIODIC TETRACHOTOMIES

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Abstract. We study the k^{th} order system of two rational difference equations

$$x_n = \frac{\beta_k x_{n-k} + \gamma_k y_{n-k}}{1 + \sum_{j=1}^{k-1} B_j x_{n-j} + \sum_{j=1}^{k-1} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{\delta_k x_{n-k} + \epsilon_k y_{n-k}}{1 + \sum_{j=1}^{k-1} D_j x_{n-j} + \sum_{j=1}^{k-1} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with nonnegative parameters and nonnegative initial conditions. We establish the existence of periodic tetrachotomy behavior which depends on the matrix

$$\begin{pmatrix} \beta_k & \gamma_k \\ \delta_k & \epsilon_k \end{pmatrix}.$$

We provide some partial results for similar systems of three or more rational difference equations.

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1. Introduction

Recently several papers discussing rational systems in the plane have appeared in the literature. We refer particularly to [2], [3], and [5]. In [2] the authors mention a conjecture regarding periodic trichotomy behavior for some rational systems in the plane. Our goal here is to try to extend a general periodic trichotomy result presented in [20] to systems of rational difference equations of order greater than one.
In [20] the author presents a broad collection of periodic trichotomy results for the k^{th} order rational difference equation with non-negative parameters and non-negative initial conditions

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, \quad n \in \mathbb{N}.$$

The author focuses on two ideas. The first idea involves expanding upon the results in [19] so as to create a general periodic trichotomy result. This idea applies in the case where $\alpha = 0$ and $A > 0$. This results in a trichotomy which depends on a comparison between $\sum_{i=1}^{k} \beta_i$ and A. The second idea is a nontrivial observation. The author observes

Date: May, 1 2010.
[0]**Keywords: difference equation, periodic convergence, systems, periodic tetrachotomy.**
AMS Subject Classification: 39A10,39A11

that there is a general trichotomy result which follows immediately from the current literature with only some minor technical work required. The author then patches together results from [4], [17], [19], [21], and [22] to create a second general periodic trichotomy result. This idea applies in the case where $\alpha > 0$ and also depends on a comparison between $\sum_{i=1}^{k} \beta_i$ and A. In this way the results presented in [20] complement each other nicely.

Let us look more closely at the first idea in [20]. The author first places conditions on the rational difference equation so that there is a nontrivial subspace of initial conditions where the solution behaves linearly. The author then shows that the rational difference equation inheirits trichotomy behavior from the associated linear difference equation.

In this paper we will build off of this idea, with the hope of creating some basic examples of trichotomy behavior for systems of rational difference equations.

2. Some Linear Algebra Review

Recall that if we have a symmetric matrix with real coefficients then such a matrix must be Hermitian. Any such matrix A is diagonalizable and has decomposition UDU^* where D is a diagonal matrix consisting of the eigenvalues of A, U is a unitary matrix, and U^* represents the conjugate transpose of U. Furthermore we know that D has only real entries. Thus we have the following facts.

Fact 1. *Suppose we have a real symmetric $m \times m$ matrix A whose spectral radius is 1 then there exists an $N \in \mathbb{N}$ so that for every $L \geq N$ we have $\langle A^L v, A^L v\rangle \leq \langle v, v\rangle$ for all $v \in \mathbb{R}^m$. Moreover $\langle A^L v, A^L v\rangle = \langle v, v\rangle$ if and only if v is in the span of the eigenvectors of A with corresponding eigenvalues whose absolute value is 1.*

Fact 2. *Suppose we have a real symmetric $m \times m$ matrix A whose spectral radius is greater than 1 and a vector $v \in \mathbb{R}^m$ so that $\langle v, w_i\rangle \neq 0$ for all $i \in \{1, \dots, m\}$, then the sequence $\{\langle A^L v, A^L v\rangle\}_{L=1}^{\infty}$ is unbounded.*

3. A Lemma

Here we present a lemma in order to simplify the results presented in the following sections.

Lemma 1. *Let V be an inner product space, let $A : V \to V$ be a bounded linear operator with respect to the inner product norm, $||\cdot||$. Assume that $\{v_n\}_{n=1}^{\infty}$ is bounded. Also assume that there exists $M > 0$ so that $||A^n v|| \leq M||v||$ for all $n \in \mathbb{N}$ and for all $v \in V$. Let $\lim_{n\to\infty} ||v_n - Av_{n-k}|| = 0$ and suppose $A^{n+m} - A^n \to 0$ for some $m \in \mathbb{N}$, then $\lim_{n\to\infty} ||v_n - v_{n-km}|| = 0$.*

Proof. For any $\rho \in \mathbb{N}$ and any $n > km + k\rho$ the triangle inequality gives us

$$||v_n - v_{n-km}|| \leq ||A^\rho v_{n-k\rho} - A^{\rho+m} v_{n-km-k\rho}|| + ||A^{\rho+m} v_{n-km-k\rho} - A^\rho v_{n-km-k\rho}|| +$$

$$||v_{n-km} - A^\rho v_{n-km-k\rho}|| + ||v_n - A^\rho v_{n-k\rho}||.$$

We want to show that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that $||v_n - v_{n-km}|| < \epsilon$ for $n \geq N$. Since $\{v_n\}_{n=1}^{\infty}$ is bounded and $A^{n+m} - A^n \to 0$, given $\epsilon > 0$ there exists $\rho_\epsilon \in \mathbb{N}$

so that $||A^{\rho_\epsilon+m}v_{n-km-k\rho_\epsilon} - A^{\rho_\epsilon}v_{n-km-k\rho_\epsilon}|| < \frac{\epsilon}{4}$ for all $n > km + k\rho_\epsilon$. For the sake of notation, from now on we let $\rho_\epsilon = \rho$. So we have

$$||v_n - v_{n-km}|| < ||A^\rho v_{n-k\rho} - A^{\rho+m}v_{n-km-k\rho}|| + \frac{\epsilon}{4} + ||v_{n-km} - A^\rho v_{n-km-k\rho}|| + ||v_n - A^\rho v_{n-k\rho}||,$$

for any $n > km + k\rho$. Notice that the triangle inequality, and the fact that $||A^n v|| \leq M||v||$ for all $n \in \mathbb{N}$ and for all $v \in V$ gives us

$$||A^\rho v_{n-k\rho} - A^{\rho+m}v_{n-km-k\rho}|| \leq \sum_{i=0}^{m-1} ||A^{\rho+i}v_{n-k(\rho+i)} - A^{\rho+i+1}v_{n-k(\rho+i+1)}||$$

$$\leq M \sum_{i=0}^{m-1} ||v_{n-k(\rho+i)} - Av_{n-k(\rho+i+1)}||,$$

for all $n > km + k\rho$. Since $\lim_{n\to\infty} ||v_n - Av_{n-k}|| = 0$, given $\epsilon > 0$ there exists an $N_1 \in \mathbb{N}$ so that $||v_n - Av_{n-k}|| < \frac{\epsilon}{4Mm}$ for $n \geq N_1$. So we get

$$||A^\rho v_{n-k\rho} - A^{\rho+m}v_{n-km-k\rho}|| < M \sum_{i=0}^{m-1} \frac{\epsilon}{4Mm} = \frac{\epsilon}{4},$$

for $n > N_1 + km + k\rho$. Thus

$$||v_n - v_{n-km}|| < \frac{\epsilon}{2} + ||v_{n-km} - A^\rho v_{n-km-k\rho}|| + ||v_n - A^\rho v_{n-k\rho}||,$$

for any $n > N_1 + km + k\rho$.
We use an argument which is similar to the prior argument. Again the triangle inequality, and the fact that $||A^n v|| \leq M||v||$ for all $n \in \mathbb{N}$ and for all $v \in V$ gives us

$$||v_n - A^\rho v_{n-k\rho}|| \leq \sum_{i=0}^{\rho-1} ||A^i v_{n-ki} - A^{i+1}v_{n-k-ki}|| \leq M \sum_{i=0}^{\rho-1} ||v_{n-ki} - Av_{n-k-ki}||,$$

for any $n > km + k\rho$. Since $\lim_{n\to\infty} ||v_n - Av_{n-k}|| = 0$, given $\epsilon > 0$ there exists an $N_2 \in \mathbb{N}$ so that $||v_n - Av_{n-k}|| < \frac{\epsilon}{4M\rho}$ for $n \geq N_2$. So we get

$$||v_n - A^\rho v_{n-k\rho}|| < M \sum_{i=0}^{\rho-1} \frac{\epsilon}{4M\rho} = \frac{\epsilon}{4},$$

for all $n > N_2 + km + k\rho$.
Again, We use a similar argument. The triangle inequality, and the fact that $||A^n v|| \leq M||v||$ for all $n \in \mathbb{N}$ and for all $v \in V$ gives us

$$||v_{n-km} - A^\rho v_{n-km-k\rho}|| \leq \sum_{i=0}^{\rho-1} ||A^i v_{n-km-ki} - A^{i+1}v_{n-k-km-ki}||$$

$$\leq M \sum_{i=0}^{\rho-1} ||v_{n-km-ki} - Av_{n-k-km-ki}||,$$

for any $n > km + k\rho$. Since $\lim_{n\to\infty} ||v_n - Av_{n-k}|| = 0$, given $\epsilon > 0$ there exists an $N_2 \in \mathbb{N}$ so that $||v_n - Av_{n-k}|| < \frac{\epsilon}{4M\rho}$ for $n \geq N_2$. So we get

$$||v_{n-km} - A^{\rho} v_{n-km-k\rho}|| < M \sum_{i=0}^{\rho-1} \frac{\epsilon}{4M\rho} = \frac{\epsilon}{4},$$

for all $n > N_2 + km + k\rho$.
So we have that given $\epsilon > 0$ there exists $N = N_1 + N_2 + km + k\rho_{\epsilon} + 1$ so that

$$||v_n - v_{n-km}|| < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon,$$

for any $n \geq N$. Thus $\lim_{n\to\infty} ||v_n - v_{n-km}|| = 0$. □

4. A Representation Using Vector Spaces

Let $k \in \mathbb{N}$, $k \neq 1$ and consider the k^{th} order system of two rational difference equations

$$x_n = \frac{\beta_k x_{n-k} + \gamma_k y_{n-k}}{1 + \sum_{j=1}^{k-1} B_j x_{n-j} + \sum_{j=1}^{k-1} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{\delta_k x_{n-k} + \epsilon_k y_{n-k}}{1 + \sum_{j=1}^{k-1} D_j x_{n-j} + \sum_{j=1}^{k-1} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with nonnegative parameters and nonnegative initial conditions. We find that it is useful to rewrite our system using matrix notation. We let

$$v_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad A = \begin{pmatrix} \beta_k & \gamma_k \\ \delta_k & \epsilon_k \end{pmatrix},$$

and

$$B_n = \begin{pmatrix} \frac{1}{1+\sum_{j=1}^{k-1} a_j \cdot v_{n-j}} & 0 \\ 0 & \frac{1}{1+\sum_{j=1}^{k-1} q_j \cdot v_{n-j}} \end{pmatrix},$$

where

$$a_j = \begin{pmatrix} B_j \\ C_J \end{pmatrix} \quad and \quad q_j = \begin{pmatrix} D_j \\ E_j \end{pmatrix}.$$

Our system then becomes

$$v_n = B_n A v_{n-k}, \quad n \in \mathbb{N}.$$

5. Some General Results

Notice that in the last section we were able to represent our k^{th} order system of two rational difference equations as a recursive system with delays on a two-dimensional vector space. In this section we prove some general results regarding recursive systems with delays on vector spaces of arbitrary finite dimension.

Theorem 1. *Consider the k^{th} order recursive system on $[0,\infty)^m$*

$$v_n = B_n A v_{n-k}, \quad n \in \mathbb{N},$$

where $A = (a_{ij})$ is a real symmetric $m \times m$ matrix with nonnegative entries $a_{ij} \geq 0$ and with initial conditions in $[0,\infty)^m$. Further assume that B_n is a real $m \times m$ diagonal matrix which may depend on n and on prior terms of our solution $\{v_n\}$, with all entries $b_{n,ij} \in [0,1]$ for all $n \in \mathbb{N}$. Then we have the following

- *I Whenever the spectral radius of A is less than 1, then every solution converges to the 0 vector.*
- *II Suppose the spectral radius of A is equal to 1 then we have $\lim_{n\to\infty} ||v_n - Av_{n-k}|| = 0$, where $||\cdot||$ represents the inner product norm.*

Proof. Let us first prove case (I). Consider the system

$$u_n = A u_{n-k}, \quad n \in \mathbb{N}.$$

Suppose $v_n = u_n$ for $n < 1$. In other words suppose that the two systems have the same initial conditions. Then the *ith* entry of the vector v_n is less than or equal to the *ith* entry of the vector u_n for all $n \in \mathbb{N}$ and for all $i \in \{1,\ldots,m\}$, in other words $v_{n,i} \leq u_{n,i}$. We prove this by strong induction on n. The initial conditions provide the base case. Suppose the result holds for $n < N$.

$$v_{N,i} = b_{N,ii} \sum_{j=1}^{m} a_{ij} v_{N-k,j} \leq \sum_{j=1}^{m} a_{ij} v_{N-k,j} \leq \sum_{j=1}^{m} a_{ij} u_{N-k,j} = u_{N,i},$$

since $b_{N,ii} \in [0,1]$ and $a_{ij} \geq 0$ for all $i,j \in \{1,\ldots,m\}$. Thus we have shown $v_{n,i} \leq u_{n,i}$ for all $n \in \mathbb{N}$.

It is clear that $u_{kn+b} = A^n u_b$. Now if the spectral radius of A is less than one it is a well known result that $\lim_{n\to\infty} A^n = 0$. Of course by 0 here we mean the zero matrix. Thus in this case $\lim_{n\to\infty} u_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. Since $v_n \in [0,\infty)^m$ for all $n \in \mathbb{N}$ we have $\lim_{n\to\infty} v_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. This finishes case (I).

Now let us consider case (II). Since A is real $m \times m$ matrix with nonnegative entries $a_{ij} \geq 0$, and A also has the property that every eigenvalue λ with $|\lambda| = 1$ has algebraic multiplicity equal to geometric multiplicity, Fact 1 applies.

Now using a similar argument as was used in case (I) we will show something slightly more general. Let $U_{n,q,L} = A^{q+L} v_{n-kL}$, we will show that $U_{n,q,0,i} \leq U_{n,q,L,i}$ whenever $L \geq 0$ for all $i \in \{1,\ldots,m\}$, $q \in \mathbb{N}$, and for all $n \in \mathbb{N}$ with $n \geq kL$. Now we will use induction on q with the case $q = 0$ providing the base case. In the case $q = 0$ we must show that $U_{n,0,0,i} \leq U_{n,0,L,i}$ for all $i \in \{1,\ldots,m\}$ and whenever $L \geq 0$ and $n \geq kL$. Consider the system

$$u_n = A u_{n-k}, \quad n \in \mathbb{N}.$$

Fix an $L \geq 0$ and fix an $N_1 \geq kL$ and suppose $v_{N_1-kL} = u_{N_1-kL}$. Then the ith entry of the vector v_{N_1} is less than or equal to the ith entry of the vector u_{N_1} for all $i \in \{1, \ldots, m\}$, in other words $v_{N_1,i} \leq u_{N_1,i}$. We prove that

$$v_{N_1-kL+kp,i} \leq u_{N_1-kL+kp,i},$$

by induction on p. The case $p = 0$ provides the base case. Suppose the result holds for $p < P$. Then we have

$$v_{N_1-kL+kP,i} = b_{N_1-kL+kP,ii} \sum_{j=1}^{m} a_{ij} v_{N_1-kL+k(P-1),j} \leq \sum_{j=1}^{m} a_{ij} v_{N_1-kL+k(P-1),j}$$

$$\leq \sum_{j=1}^{m} a_{ij} u_{N_1-kL+k(P-1),j} = u_{N_1-kL+kP,i},$$

since $b_{N_1-kL+kP,ii} \in [0,1]$ and $a_{ij} \geq 0$ for all $i,j \in \{1,\ldots,m\}$. Thus we have shown in this case that $v_{N_1,i} \leq u_{N_1,i}$ for all $i \in \{1,\ldots,m\}$. Notice that $v_{N_1,i} = U_{N_1,0,0,i}$ and $u_{N_1,i} = U_{N_1,0,L,i}$. Further notice that this is true for any fixed $L \geq 0$ and fixed $N_1 \geq kL$. Thus $U_{n,0,0,i} \leq U_{n,0,L,i}$ for all $i \in \{1,\ldots,m\}$ and whenever $L \geq 0$ and $n \geq kL$. So we have shown the case $q = 0$.
Now we use the case $q = 0$ as a base case in an induction argument on q. Suppose that $U_{n,q,0,i} \leq U_{n,q,L,i}$, then since all entries a_{ij} of our matrix A are non-negative and $U_{n,q,L}$ is a vector with all non-negative entries,

$$U_{n,q+1,0,i} = \sum_{j=1}^{m} a_{ij} U_{n,q,0,j} \leq \sum_{j=1}^{m} a_{ij} U_{n,q,L,j} = U_{n,q+1,L,i}.$$

For shorthand we define a function $h : \mathbb{R}^m \to [0,\infty)$ so that $h(v) = \langle v, v \rangle$. Fact 1 gives us that there exists an $N \in \mathbb{N}$ so that $h(A^L v) \leq h(v)$ for all $v \in \mathbb{R}^m$ and all $L \geq N$. Thus fixing an $L \geq N$ and a $q \in \mathbb{N}$ we have $h(A^q v_n) \leq h(A^{L+q} v_{n-kL}) \leq h(A^q v_{n-kL})$. Since each of the subsequences $\{h(A^q v_{nkL+a})\}_{n=1}^{\infty}$ are monotone decreasing and bounded below by zero, they all converge. So $\lim_{n\to\infty} h(A^q v_n) - h(A^q v_{n-kL}) = 0$. By the squeeze theorem we get $\lim_{n\to\infty} h(A^q v_n) - h(A^{L+q} v_{n-kL}) = 0$. We use the triangle inequality and we get $|h(v_n) - h(Av_{n-k})| \leq |h(A^{L_1+1} v_{n-k-kL_1}) - h(Av_{n-k})| + |h(A^{L_1+1} v_{n-k-kL_1}) - h(v_n)|$. Since, $\lim_{n\to\infty} h(A^q v_n) - h(A^{L+q} v_{n-kL}) = 0$, we may substitute first with $L = L_1 > N$ and $q = 1$. This shows us that $\lim_{n\to\infty} |h(A^{L_1+1} v_{n-k-kL_1}) - h(Av_{n-k})| = 0$. We may also substitute $L = L_1 + 1$ and $q = 0$ to see that $\lim_{n\to\infty} |h(A^{L_1+1} v_{n-k-kL_1}) - h(v_n)| = 0$. This implies $\lim_{n\to\infty} h(v_n) - h(Av_{n-k}) = 0$.
Using the notation we have established earlier we may rewrite Av_{n-k} as $U_{n,0,1}$ and we may write $v_n = B_n U_{n,0,1}$. Recall that B_n is a diagonal matrix which may depend on n and on prior terms of our solution $\{v_n\}$, with all entries $b_{n,ij} \in [0,1]$ for all $n \in \mathbb{N}$. Thus since B_n is a diagonal matrix we have

$$\lim_{n\to\infty} \sum_{i=1}^{m} (b_{n,ii}^2 - 1) U_{n,0,1,i}^2 = 0.$$

Since all entries $b_{n,ij} \in [0,1]$ for all $n \in \mathbb{N}$ and all entries of $U_{n,0,1}$ are nonnegative for all $n \in \mathbb{N}$ we have that $(b_{n,ii}^2 - 1)U_{n,0,1,i}^2 \leq 0$ for all $i \in \{1, \ldots, m\}$. Thus

$$\lim_{n\to\infty} (b_{n,ii}^2 - 1)U_{n,0,1,i}^2 = 0,$$

for all $i \in \{1, \ldots, m\}$. Thus

$$\lim_{n\to\infty} (1 - b_{n,ii}^2)U_{n,0,1,i}^2 = 0,$$

for all $i \in \{1, \ldots, m\}$. Since $b_{n,ij} \in [0,1]$ for all $n \in \mathbb{N}$ we get,

$$b_{n,ii}^2 - 1 \leq (b_{n,ii} - 1)^2 \leq 1 - b_{n,ii}^2,$$

for all $n \in \mathbb{N}$. This gives us

$$\lim_{n\to\infty} (b_{n,ii} - 1)U_{n,0,1,i} = 0,$$

for all $i \in \{1, \ldots, m\}$. This implies that

$$\lim_{n\to\infty} v_n - Av_{n-k} = 0.$$

So

$$\lim_{n\to\infty} ||v_n - Av_{n-k}|| = 0.$$

This completes the proof of the theorem. □

Theorem 2. *Consider the k^{th} order recursive system on $[0,\infty)^m$*

$$v_n = B_n A v_{n-k}, \quad n \in \mathbb{N},$$

where $A = (a_{ij})$ is a real symmetric $m \times m$ matrix with nonnegative entries $a_{ij} \geq 0$ and with initial conditions in $[0,\infty)^m$. Further assume that B_n is a real $m \times m$ diagonal matrix with entries $b_{n,ii} = \frac{1}{1+\sum_{j=1}^{k-1} q_{ij} \cdot v_{n-j}}$ for all $n \in \mathbb{N}$. Where the vectors $q_{ij} \in [0,\infty)^m$. If the spectral radius of A is greater than 1. Then for some choice of initial conditions the solution $\{v_n\}_{n=1}^{\infty}$ is such that $\{||v_n||\}_{n=1}^{\infty}$ is an unbounded sequence.

Proof. Before we begin to prove the first case notice that if we choose initial conditions so that $v_n = 0$ for $n < 1$ and $n \neq 1-k$, then it is clear by a simple induction argument that $v_n = 0$ for $n \not\equiv 1 \mod k$. Thus for solutions with these initial conditions we have $v_n = Av_{n-k}$. So here we intend to take advantage of this linearity so we will assume that $v_n = 0$ for $n < 1$ and $n \neq 1-k$, and our goal will be to choose v_{1-k} appropriately in order to create an unbounded solution.

If we choose $v_{1-k} \in [0,\infty)^m$ so that for all the eigenvectors of A, $w_1, \ldots w_m$, $\langle v_{1-k}, w_i \rangle \neq 0$ for all $i \in \{1, \ldots, m\}$, this is certainly possible since $[0,\infty)^m$ is an m-dimensional subspace of $\mathbb{R}^m$. We apply Fact 2 and we notice that $||v_{kL+1}|| = ||A^{L+1}v_{1-k}|| = \sqrt{\langle A^{L+1}v_{1-k}, A^{L+1}v_{1-k} \rangle}$, thus $\{||v_{kL+1}||\}_{L=1}^{\infty}$ is unbounded, so $\{||v_n||\}_{n=1}^{\infty}$ is unbounded. □

6. A General Periodic Trichotomy Result

In this section we use the Perron-Frobenius theorem along with our work in the last section to demonstrate a general periodic trichotomy result. For more details regarding the Perron-Frobenius theorem see [10] chapter 8 sections 2 and 3.

Theorem 3. *Consider the k^{th} order recursive system on $[0,\infty)^m$*

$$v_n = B_n A v_{n-k}, \quad n \in \mathbb{N},$$

where $A = (a_{ij})$ is a real symmetric $m \times m$ matrix with positive entries $a_{ij} > 0$ and with initial conditions in $[0,\infty)^m$. Further assume that B_n is a real $m \times m$ diagonal matrix with entries $b_{n,ii} = \frac{1}{1+\sum_{j=1}^{k-1} q_{ij}\cdot v_{n-j}}$ for all $n \in \mathbb{N}$. Where the vectors $q_{ij} \in [0,\infty)^m$. Then this system displays the following trichotomy behavior:

- *i If the spectral radius of A is less than 1 then every solution converges to the 0 vector.*
- *ii If the spectral radius of A is equal to 1 then every solution converges to a solution of not necessarily prime period k. Furthermore in this case there exist solutions of prime period k.*
- *iii If the spectral radius of A is greater than 1 then for some choice of initial conditions the solution $\{v_n\}_{n=1}^{\infty}$ has the property that $\{||v_n||\}_{n=1}^{\infty}$ is an unbounded sequence. Moreover if we consider the sequences consisting of the entries of v_n, $\{v_{n,i}\}_{n=1}^{\infty}$, then $\{v_{n,i}\}_{n=1}^{\infty}$ is an unbounded sequence for every $i \in \{1,\dots,m\}$.*

Proof. First notice that (i) follows immediately from Theorem 1. Now consider case (iii). From Theorem 2 we get immediately that there is some choice of initial conditions so that the solution $\{v_n\}_{n=1}^{\infty}$ has the property that $\{||v_n||\}_{n=1}^{\infty}$ is an unbounded sequence. Recall from the proof of Theorem 2 that every unbounded solution we constructed had the property that $v_n = 0$ for $n < 1$ and $n \neq 1-k$. For our purposes we will choose an unbounded solution which has this property, thus $v_n = Av_{n-k}$ for our solution. Since $\{||v_n||\}_{n=1}^{\infty}$ is an unbounded sequence it follows as a consequence $\{v_{n,i_1}\}_{n=1}^{\infty}$ is an unbounded sequence for some $i_1 \in \{1,\dots,m\}$. So there is a subsequence $\{v_{n_L,i_1}\}$ which diverges to ∞. Since $A = (a_{ij})$ is a real $m \times m$ matrix with positive entries $a_{ij} > 0$ and $v_{n_L+k} = Av_{n_L}$, the subsequence $\{v_{n_L+k,i}\}$ diverges to ∞ for all $i \in \{1,\dots,m\}$. So $\{v_{n,i}\}_{n=1}^{\infty}$ is an unbounded sequence for all $i \in \{1,\dots,m\}$. This concludes the proof of case (iii).

To prove case (ii) we use the Perron-Frobenius theorem. The Perron-Frobenius theorem tells us that if $A = (a_{ij})$ is a real $m \times m$ matrix with positive entries $a_{ij} > 0$, then there is a positive real number r called the Perron-Frobenius eigenvalue such that r is an eigenvalue of A and so that any other possibly complex eigenvalue λ has $|\lambda| < r$. Moreover r is a simple root of the characteristic polynomial and there is an eigenvector w_r associated with r having strictly positive components. Now combining this with the fact that the spectral radius is 1 we get that $r = 1$ and every other eigenvalue λ has $|\lambda| < 1$. Also we know that r is a simple root of the characteristic polynomial so r has algebraic multiplicity equal to 1. So it must be true that every eigenvalue λ with $|\lambda| = 1$ has algebraic multiplicity equal to geometric multiplicity. Thus Fact 1 applies in this case.

The next part of this argument will be almost identical to the argument presented in Theorem 1. Let $U_{n,q,L} = A^{q+L}v_{n-kL}$, we will show that $U_{n,q,0,i} \leq U_{n,q,L,i}$ whenever $L \geq 0$ for all $i \in \{1, \dots, m\}$, $q \in \mathbb{N}$, and for all $n \in \mathbb{N}$ with $n \geq kL$. Now we will use induction on q with the case $q = 0$ providing the base case. In the case $q = 0$ we must show that $U_{n,0,0,i} \leq U_{n,0,L,i}$ for all $i \in \{1, \dots, m\}$ and whenever $L \geq 0$ and $n \geq kL$. Consider the system

$$u_n = Au_{n-k}, \quad n \in \mathbb{N}.$$

Fix an $L \geq 0$ and fix an $N_1 \geq kL$ and suppose $v_{N_1-kL} = u_{N_1-kL}$. Then the ith entry of the vector v_{N_1} is less than or equal to the ith entry of the vector u_{N_1} for all $i \in \{1, \dots, m\}$, in other words $v_{N_1,i} \leq u_{N_1,i}$. We prove that

$$v_{N_1-kL+kp,i} \leq u_{N_1-kL+kp,i},$$

by induction on p. The case $p = 0$ provides the base case. Suppose the result holds for $p < P$. Then we have

$$v_{N_1-kL+kP,i} = b_{N_1-kL+kP,ii} \sum_{j=1}^{m} a_{ij} v_{N_1-kL+k(P-1),j} \leq \sum_{j=1}^{m} a_{ij} v_{N_1-kL+k(P-1),j}$$

$$\leq \sum_{j=1}^{m} a_{ij} u_{N_1-kL+k(P-1),j} = u_{N_1-kL+kP,i},$$

since $b_{N_1-kL+kP,ii} \in [0, 1]$ and $a_{ij} \geq 0$ for all $i, j \in \{1, \dots, m\}$. Thus we have shown in this case that $v_{N_1,i} \leq u_{N_1,i}$ for all $i \in \{1, \dots, m\}$. Notice that $v_{N_1,i} = U_{N_1,0,0,i}$ and $u_{N_1,i} = U_{N_1,0,L,i}$. Further notice that this is true for any fixed $L \geq 0$ and fixed $N_1 \geq kL$. Thus $U_{n,0,0,i} \leq U_{n,0,L,i}$ for all $i \in \{1, \dots, m\}$ and whenever $L \geq 0$ and $n \geq kL$. So we have shown the case $q = 0$.
Now we use the case $q = 0$ as a base case in an induction argument on q. Suppose that $U_{n,q,0,i} \leq U_{n,q,L,i}$, then since all entries a_{ij} of our matrix A are non-negative and $U_{n,q,L}$ is a vector with all non-negative entries,

$$U_{n,q+1,0,i} = \sum_{j=1}^{m} a_{ij} U_{n,q,0,j} \leq \sum_{j=1}^{m} a_{ij} U_{n,q,L,j} = U_{n,q+1,L,i}.$$

For shorthand we define a function $h : \mathbb{R}^m \to [0, \infty)$ so that $h(v) = \langle v, v \rangle$. Fact 1 gives us that there exists an $N \in \mathbb{N}$ so that $h(A^L v) \leq h(v)$ for all $v \in \mathbb{R}^m$ and all $L \geq N$. Thus fixing an $L \geq N$ and a $q \in \mathbb{N}$ we have $h(A^q v_n) \leq h(A^{L+q} v_{n-kL}) \leq h(A^q v_{n-kL})$. Since each of the subsequences $\{h(A^q v_{nkL+a})\}_{n=1}^{\infty}$ are monotone decreasing and bounded below by zero, they all converge. So $\lim_{n\to\infty} h(A^q v_n) - h(A^q v_{n-kL}) = 0$. By the squeeze theorem we get $\lim_{n\to\infty} h(A^q v_n) - h(A^{L+q} v_{n-kL}) = 0$.
So the subsequences $\{h(v_{nkL+a})\}_{n=1}^{\infty}$ and $\{h(A^L v_{nkL+a})\}_{n=1}^{\infty}$ are convergent and since $\lim_{n\to\infty} h(v_n) - h(A^L v_{n-kL}) = 0$ we get

$$\lim_{n\to\infty} h(v_{nkL+a}) = \mathfrak{L}_a = \lim_{n\to\infty} h(A^L v_{nkL+a}).$$

Now consider the sequence $\{v_{nkL+a}\}_{n=1}^{\infty}$ and let $\{v_{n_j kL+a}\}_{j=1}^{\infty}$ be a convergent subsequence with $\lim_{j\to\infty} v_{n_j kL+a} = w_a$. By what we have just shown it must be true that $h(w_a) =$

$h(A^L w_a)$, but then by Fact 1 we have that w_a is in the span of the eigenvectors of A with corresponding eigenvalues whose absolute value is 1. Recall from the Perron-Frobenius theorem that there is only one such eigenvector and it is w_1, the eigenvector associated to the eigenvalue 1. So $w_a = cw_1$, where c is an arbitrary constant, and $h(w_a) = \mathfrak{L}_a$, also $w_a \in [0,\infty)^m$ as a consequence of our choice of initial conditions. Thus $w_a = w_1\left(\frac{\sqrt{\mathfrak{L}_a}}{||w_1||}\right)$. What this means is that the sequence $\{v_{nkL+a}\}_{n=1}^{\infty}$ must converge to $w_a = w_1\left(\frac{\sqrt{\mathfrak{L}_a}}{||w_1||}\right)$. Suppose it does not, then for some $\epsilon > 0$ there is a subsequence $\{v_{n_dkL+a}\}_{d=1}^{\infty}$ so that

$$||v_{n_dkL+a} - w_1\left(\frac{\sqrt{\mathfrak{L}_a}}{||w_1||}\right)|| > \epsilon$$

for all $d \in \mathbb{N}$. However we know that $\{v_{n_dkL+a}\}_{d=1}^{\infty}$ is bounded and so it has a convergent subsequence. This means that $\{v_{nkL+a}\}_{n=1}^{\infty}$ has a convergent subsequence which does not converge to $w_1\left(\frac{\sqrt{\mathfrak{L}_a}}{||w_1||}\right)$. We have already shown that every convergent subsequence of $\{v_{nkL+a}\}_{n=1}^{\infty}$ converges to $w_1\left(\frac{\sqrt{\mathfrak{L}_a}}{||w_1||}\right)$. Thus we have a contradiction. This proves that the sequence $\{v_{nkL+a}\}_{n=1}^{\infty}$ must converge to $w_1\left(\frac{\sqrt{\mathfrak{L}_a}}{||w_1||}\right)$.
Now we want to apply Theorem 1 and Lemma 1. It is clear from the linear algebra discussion earlier that the sequence of matrices $\{D^n\}_{n=1}^{\infty}$ converges to a diagonal matrix with diagonal entries either 1 or 0. Since $\{D^n\}_{n=1}^{\infty}$ converges, $\{A^n\}_{n=1}^{\infty}$ converges. Thus $A^{n+1} - A^n \to 0$. Also it is clear from this that $\{v_n\}_{n=1}^{\infty}$ is bounded and that there exists $M > 0$ so that $||A^n v|| \leq M||v||$ for all $n \in \mathbb{N}$ and for all $v \in V$. Now we use Theorem 1 to show that $\lim_{n\to\infty} ||v_n - Av_{n-k}|| = 0$. So we apply Lemma 1 and we get that $\lim_{n\to\infty} ||v_n - v_{n-k}|| = 0$.
Combining the fact that $\{v_{nkL+a}\}_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} ||v_n - v_{n-k}|| = 0$ we get that if $a_1 \equiv a_2 \mod k$ then

$$\lim_{n\to\infty} v_{nkL+a_1} = \lim_{n\to\infty} v_{nkL+a_2}.$$

Thus every solution must converge to a periodic solution of not necessarily prime period k.

To construct a solution which is periodic with prime period k we use our eigenvector w_1 associated with the eigenvalue 1 having strictly positive components. We choose initial conditions so that $v_n = 0$ for $n < 1$ and $n \neq 1-k$, and $v_{1-k} = w_1$. This is a periodic solution of prime period k. This concludes our proof. □

7. A Periodic Tetrachotomy Result

Now we combine all of our work to give some preliminary examples of periodic tetrachotomy behavior for systems of two rational difference equations.

Theorem 4. *Let $k \in \mathbb{N}$, $k \neq 1$ and consider the k^{th} order system of two rational difference equations*

$$x_n = \frac{\beta_k x_{n-k} + \gamma_k y_{n-k}}{1 + \sum_{j=1}^{k-1} B_j x_{n-j} + \sum_{j=1}^{k-1} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{\delta_k x_{n-k} + \epsilon_k y_{n-k}}{1 + \sum_{j=1}^{k-1} D_j x_{n-j} + \sum_{j=1}^{k-1} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with nonnegative parameters and nonnegative initial conditions. Define a matrix

$$A = \begin{pmatrix} \beta_k & \gamma_k \\ \delta_k & \epsilon_k \end{pmatrix}.$$

Assume $\delta_k = \gamma_k$, then this system exhibits the following tetrachotomy behavior.

- *I Suppose the spectral radius of A is less than 1, then every solution converges to the unique equilibrium.*
- *II Suppose the spectral radius of A is equal to 1 and -1 is not an eigenvalue of A, then every solution converges to a periodic solution of not necessarily prime period k. Furthermore in this case there exist periodic solutions with prime period k.*
- *III Suppose the spectral radius of A is equal to 1 and -1 is an eigenvalue of A, then every solution converges to a periodic solution of not necessarily prime period $2k$. Furthermore in this case there exist periodic solutions with prime period $2k$.*
- *IV Suppose the spectral radius of A is greater than 1 then there exist solutions where $x_n + y_n$ is unbounded.*

Proof. To begin we rewrite our system using matrix notation, as was done in Section 4. We let

$$v_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad A = \begin{pmatrix} \beta_k & \gamma_k \\ \delta_k & \epsilon_k \end{pmatrix},$$

and

$$B_n = \begin{pmatrix} \frac{1}{1+\sum_{j=1}^{k-1} a_j \cdot v_{n-j}} & 0 \\ 0 & \frac{1}{1+\sum_{j=1}^{k-1} q_j \cdot v_{n-j}} \end{pmatrix},$$

where

$$a_j = \begin{pmatrix} B_j \\ C_J \end{pmatrix} \quad \textit{and} \quad q_j = \begin{pmatrix} D_j \\ E_j \end{pmatrix}.$$

Our system then becomes

$$v_n = B_n A v_{n-k}, \quad n \in \mathbb{N}.$$

Now case (I) follows directly from Theorem 1. Also case (IV) follows directly from Theorem 2. In case (II) the subcase where $\beta_k, \gamma_k, \delta_k, \epsilon_k > 0$ is covered by Theorem 3. Thus all that remains to prove is case (II) in the subcase where one of the constants is 0 and case (III).

Recall that the solutions for the eigenvalues of a 2×2 matrix A can be written as

$$\lambda = \frac{1}{2}\left(tr(A) \pm \sqrt{tr^2(A) - 4det(A)}\right).$$

This computation is fairly straightforward; it appears as an exercise on page 39 in [15]. With our definition of A this becomes

$$\lambda = \frac{1}{2}\left(\beta_k + \epsilon_k \pm \sqrt{(\beta_k - \epsilon_k)^2 + 4\gamma_k\delta_k}\right).$$

we may write A in Jordan normal form and we get

$$A = W \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} W^{-1}.$$

For the sake of notation we will sometimes use

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Now consider a function $h : \mathbb{R}^2 \to [0, \infty)$ where $h(v) = \langle v, v\rangle$ the dot product of v with itself. Let w_1 and w_2 be the unit eigenvectors of A, since A is diagonalizable w_1 and w_2 form an orthonormal basis for $\mathbb{R}^2$. It follows that for all $v \in \mathbb{R}^2$,

$$h(Av) = \langle Av, Av\rangle = \sum_{i=1}^{2} |\langle Av, w_i\rangle|^2 = \sum_{i=1}^{2} |\langle WDW^{-1}v, w_i\rangle|^2$$

$$= \sum_{i=1}^{2} |\lambda_i|^2 |\langle v, w_i\rangle|^2 \leq \sum_{i=1}^{2} |\langle v, w_i\rangle|^2 = \langle v, v\rangle = h(v).$$

Also it is clear here that if $h(Av) = h(v)$ and $|\lambda_i| < 1$ for some $i \in \{1, 2\}$ then $\langle v, w_i\rangle = 0$. In other words Fact 1 applies here, and since A is diagonalizable it applies for $L \geq N = 1$. Let $U_n = Av_{n-k}$, then we have

$$v_{n,1} = \frac{\beta_k v_{n-k,1} + \gamma_k v_{n-k,2}}{1 + \sum_{j=1}^{k-1} a_j \cdot v_{n-j}} \leq \beta_k v_{n-k,1} + \gamma_k v_{n-k,2} = U_{n,1}$$

since $a_j, v_{n-j} \in [0, \infty)^2$. Also we have

$$v_{n,2} = \frac{\delta_k v_{n-k,1} + \epsilon_k v_{n-k,2}}{1 + \sum_{j=1}^{k-1} q_j \cdot v_{n-j}} \leq \delta_k v_{n-k,1} + \epsilon_k v_{n-k,2} = U_{n,2}$$

since $q_j, v_{n-j} \in [0, \infty)^2$. So $v_{n,i} \leq U_{n,i}$ for all $i \in \{1, 2\}$ where $U_n = Av_{n-k}$. Thus $h(Av_n) \leq h(v_n) \leq h(Av_{n-k}) \leq h(v_{n-k})$. Since each of the subsequences $\{h(v_{nk+a})\}_{n=1}^{\infty}$ and $\{h(Av_{nk+a})\}_{n=1}^{\infty}$ are monotone decreasing and bounded below by 0, they all converge. So $\lim_{n\to\infty} h(v_n) - h(v_{n-k}) = 0$. By the squeeze theorem we get $\lim_{n\to\infty} h(v_n) - h(Av_{n-k}) = 0$. Thus we have

$$\lim_{n\to\infty} h(v_{nk+a}) = \mathfrak{L}_a = \lim_{n\to\infty} h(Av_{nk+a}).$$

Now to finish case (II) our argument must break into several cases.
Recall that in case (II) our eigenvalues must be real and not equal to -1 and must have the property that $|\lambda_i| \leq 1$ since the spectral radius is 1 in this case. So we have two options, both λ_1 and λ_2 are equal to 1, or one of the two eigenvalues λ_i has the property that $|\lambda_i| < 1$. In the case where one of the two eigenvalues λ_i has the property that $|\lambda_i| < 1$ we will use an argument almost identical to the argument which we applied in Theorem 3. If we are in this case one of the two eigenvalues, call it λ_2, has the property that $|\lambda_2| < 1$. If it is actually λ_1 with $|\lambda_1| < 1$ then the argument is similar and will be omitted. Now consider the sequence $\{v_{nk+a}\}_{n=1}^{\infty}$ and let $\{v_{n_jk+a}\}_{j=1}^{\infty}$ be a convergent subsequence with $\lim_{j\to\infty} v_{n_jk+a} = w_a$. By what we have just shown it must be true in

this case that $h(w_a) = h(Aw_a)$, but then by Fact 1 we have that w_a is in the span of the eigenvectors of A with corresponding eigenvalues whose absolute value is 1. Notice that in this case there is only one such eigenvector and it is w_1, the eigenvector associated to the eigenvalue 1. So $w_a = cw_1$, where c is an arbitrary constant, and $h(w_a) = \mathfrak{L}_a$, also $w_a \in [0,\infty)^m$ as a consequence of our choice of initial conditions. Thus $w_a = w_1\left(\frac{\sqrt{\mathfrak{L}_a}}{||w_1||}\right)$. What this means is that the sequence $\{v_{nk+a}\}_{n=1}^{\infty}$ must converge to $w_a = w_1\left(\frac{\sqrt{\mathfrak{L}_a}}{||w_1||}\right)$. Suppose it does not, then for some $\epsilon > 0$ there is a subsequence $\{v_{n_dk+a}\}_{d=1}^{\infty}$ so that

$$||v_{n_dk+a} - w_1\left(\frac{\sqrt{\mathfrak{L}_a}}{||w_1||}\right)|| > \epsilon$$

for all $d \in \mathbb{N}$. However we know that $\{v_{n_dk+a}\}_{d=1}^{\infty}$ is bounded and so it has a convergent subsequence. This means that $\{v_{nk+a}\}_{n=1}^{\infty}$ has a convergent subsequence which does not converge to $w_1\left(\frac{\sqrt{\mathfrak{L}_a}}{||w_1||}\right)$. We have already shown that every convergent subsequence of $\{v_{nk+a}\}_{n=1}^{\infty}$ converges to $w_1\left(\frac{\sqrt{\mathfrak{L}_a}}{||w_1||}\right)$. Thus we have a contradiction. This proves that the sequence $\{v_{nk+a}\}_{n=1}^{\infty}$ must converge to $w_1\left(\frac{\sqrt{\mathfrak{L}_a}}{||w_1||}\right)$. Thus in this case every solution must converge to a periodic solution of not necessarily prime period k.

Now let us consider the other option in case (II). Assume that we have $\delta_k = \gamma_k = 0$ in case (II). Then we have for $0 < \lambda < 1$, $A = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, or $A = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, Let us focus on the recursive equations for x_n and y_n, we get that

$$x_n = \frac{x_{n-k}}{1 + \sum_{j=1}^{k-1} B_j x_{n-j} + \sum_{j=1}^{k-1} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{\lambda y_{n-k}}{1 + \sum_{j=1}^{k-1} D_j x_{n-j} + \sum_{j=1}^{k-1} E_j y_{n-j}}, \quad n \in \mathbb{N}.$$

So we obtain the following recursive inequalities

$$x_n \le x_{n-k}, \quad n \in \mathbb{N},$$

$$y_n \le \lambda y_{n-k}, \quad n \in \mathbb{N}.$$

So the subsequences $\{x_{nk+a}\}_{n=1}^{\infty}$ and $\{y_{nk+a}\}_{n=1}^{\infty}$ are all monotone decreasing and bounded below by zero, so they all converge and clearly $y_n \to 0$.

Or we have

$$x_n = \frac{\lambda x_{n-k}}{1 + \sum_{j=1}^{k-1} B_j x_{n-j} + \sum_{j=1}^{k-1} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{y_{n-k}}{1 + \sum_{j=1}^{k-1} D_j x_{n-j} + \sum_{j=1}^{k-1} E_j y_{n-j}}, \quad n \in \mathbb{N}.$$

So we obtain the following recursive inequalities

$$x_n \le \lambda x_{n-k}, \quad n \in \mathbb{N},$$

$$y_n \le y_{n-k}, \quad n \in \mathbb{N}.$$

So the subsequences $\{x_{nk+a}\}_{n=1}^{\infty}$ and $\{y_{nk+a}\}_{n=1}^{\infty}$ are all monotone decreasing and bounded below by zero, so they all converge and clearly $x_n \to 0$. To construct a periodic solution take the initial conditions to be $v_n = 0$ for $n < 1$ and $n \neq 1-k$ and $v_{1-k} = (0,1)$ or $v_{1-k} = (1,0)$ depending on the case. This is a periodic solution of prime period k. This concludes the proof of case (II).

Now we will prove case (III). Now suppose $\beta_k + \epsilon_k > 0$ and -1 is an eigenvalue. Then we must have

$$\frac{1}{2}\left(\beta_k + \epsilon_k + \sqrt{(\beta_k - \epsilon_k)^2 + 4\gamma_k\delta_k}\right) > 1.$$

However since we have assumed that the spectral radius is 1 in this case that is impossible. Thus $\beta_k + \epsilon_k \leq 0$ and we know from assumption that $\beta_k + \epsilon_k \geq 0$. Thus $\beta_k + \epsilon_k = 0$ and in case (III) both -1 and 1 are eigenvalues. So in case (III) we have

$$A = \left(\begin{array}{cc} 0 & \gamma_k \\ \frac{1}{\gamma_k} & 0 \end{array}\right).$$

So in case (III) we have the following system of rational difference equations

$$x_n = \frac{\gamma_k y_{n-k}}{1 + \sum_{j=1}^{k-1} B_j x_{n-j} + \sum_{j=1}^{k-1} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{x_{n-k}}{\gamma_k(1 + \sum_{j=1}^{k-1} D_j x_{n-j} + \sum_{j=1}^{k-1} E_j y_{n-j})}, \quad n \in \mathbb{N}.$$

Thus we have the following recursive inequalities

$$x_n \leq x_{n-2k},$$

$$y_n \leq y_{n-2k}.$$

So the subsequences $\{y_{n2k+a}\}_{n=1}^{\infty}$ and $\{x_{n2k+a}\}_{n=1}^{\infty}$ are all monotone decreasing and bounded below by zero, so they all converge. Thus we have shown that in case (III) every solution converges to a periodic solution of not necessarily prime period $2k$. Since in case (III) we have

$$A = \left(\begin{array}{cc} 0 & \gamma_k \\ \frac{1}{\gamma_k} & 0 \end{array}\right),$$

choose initial conditions where $v_n = 0$ for $n < 1$ and $n \neq 1-k$ and

$$v_{1-k} = \left(\begin{array}{c} a \\ b \end{array}\right),$$

where $a, b \in [0, \infty)$ and $a \neq \gamma_k b$ and the solution given by these initial conditions is a periodic solution of prime period $2k$. This concludes the proof of case (III).

□

8. Conclusion

We have created some analogues for trichotomy behavior for systems of rational difference equations, but we have barely scratched the surface. There are literally thousands of special cases of systems of rational difference equations of order greater than one to explore. This paper leaves several questions for further study. Are there any other examples of periodic tetrachotomy behavior for systems of two rational difference equations? Is it possible to make analogues to other trichotomy results in the literature? The main idea to take away from this article is that in some cases it is useful to reframe a problem about systems of rational difference equations as a problem about recursive systems on vector spaces. Doing this allows one to utilize the powerful tools available in linear algebra.

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